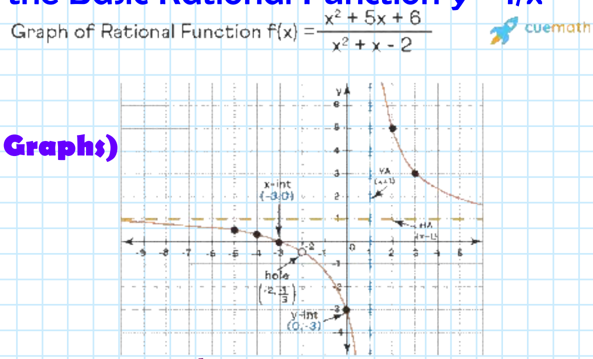
Thursday, Nov. 23rd

Plan For Today:

1. Go over Unit 3 Exam
    - Rewrite Unit 3 Exam after class Tuesday at 12:30pm
2. Any questions from 9.1 Graphing Transformations on the Basic Rational Function y = 1/x from last class?



3. Continue Chapter 9: Rational Functions
    - ✓ 9.1: Rational Function Transformations
    - **9.2: Analysing Rational Functions (Characteristics of Graphs)**
    - 9.3: Graphs and Solving Rational Functions
4. Work on Practice Questions from Workbook

Plan Going Forward:

1. Finish going through 9.2 questions in workbook and continue working on practice review handout.

    - **CHECK-IN QUIZ ON 9.2 ON TUESDAY, NOV. 28TH**

    - CHAPTER 9 PROJECT DUE TUESDAY, DEC. 5TH
    - CHAPTER 9 QUIZ ON TUESDAY, DEC. 5TH

2. We will continue Chapter 9 on Thursday.

    - UNIT 3 EXAM **REWRITE** ON TUESDAY, NOV. 28TH
        - Start time is 12:30pm

Please let me know if you have any questions or concerns about your progress in this course.
The notes from today will be posted at anurita.weebly.com after class.
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Chapter 9 Rational Functions

Do #1,2,4,5 Practice

9.1 Exploring Rational Functions Using Transformations

KEY IDEAS

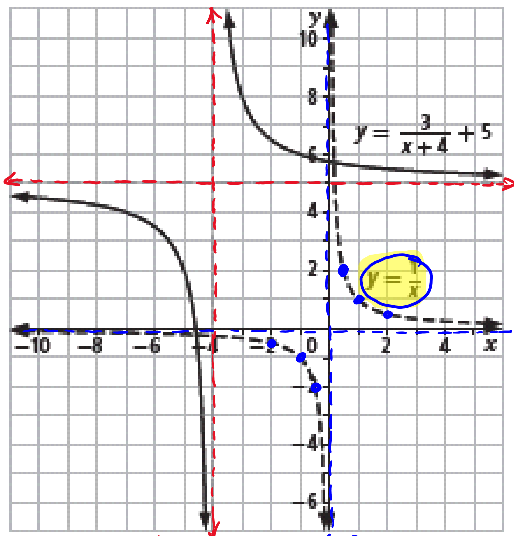
- Rational functions are functions of the form $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial expressions and $q(x) \neq 0$.
- You can graph a rational function by creating a table of values and then graphing the points in the table. To create a table of values,
  - identify the non-permissible value(s)
  - write the non-permissible value in the middle row of the table
  - enter positive values above the non-permissible value and negative values below the non-permissible value
  - choose small and large values of x to give you a spread of values
- You can use what you know about the base function $y = \frac{1}{x}$ and transformations to graph equations of the form $\boxed{y = \frac{a}{x-h} + k}$. or $y = \frac{a}{b(x-h)} + k$

  combine to equal 'a' only

  Example:

  For $y = \frac{3}{x+4} + 5$, the values of the parameters are

  VE@3 $a = 3$, representing a vertical stretch by a factor of 3
  $h = 4$, representing a horizontal translation 4 units to the left
  $k = 5$, representing a vertical translation 5 units up
  vertical asymptote: $x = -4$
  horizontal asymptotes: $y = 5$



y=5 (HA)

(VA) $x=-4$

$x=0$

$y=0$

$y=\frac{1}{x}$

x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	2
0	∅
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

- Some equations of rational functions can be manipulated algebraically into the form $y = \frac{a}{x-h} + k$ by creating a common factor in the numerator and the denominator.

  Example:

$$y = \frac{3x+6}{x-4}$$

$$y = \frac{3x-12+12+6}{x-4}$$

$$y = \frac{3x-12+18}{x-4}$$

$$y = \frac{3(x-4)}{x-4} + \frac{18}{x-4}$$

$$y = \frac{18}{x-4} + 3$$

skip

$y = \frac{18}{x-4} + 3$

Working Example 1: Graph a Rational Function Using a Table of Values

Graph $y = \frac{4}{x}$ using a table of values.

Solution

VE ≠ 4

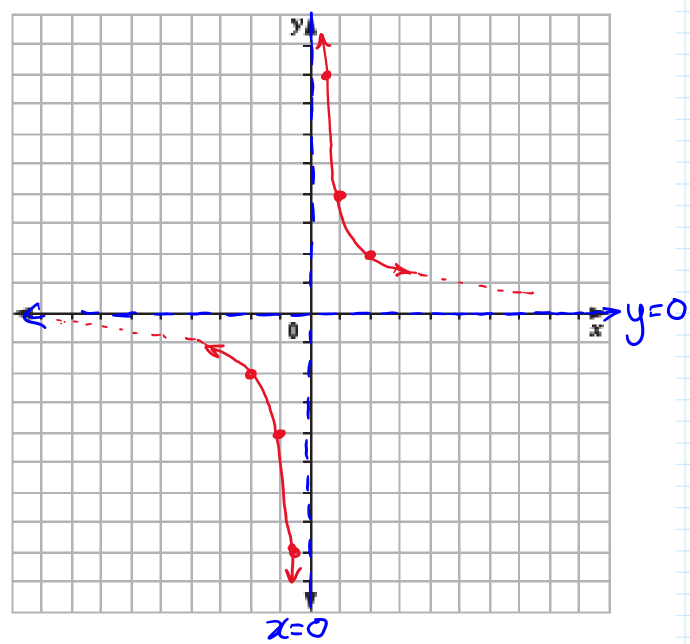
Begin by identifying any non-permissible values: what value(s) can x not equal? $x \neq$ 0.

Graphs of rational functions of the form $y = \frac{4}{x}$ approach asymptotes at $x =$ 0 and $y =$ 0. Plot the vertical and horizontal asymptotes on the grid below.

Create a table of values. Plot and connect the points from the table of values to generate the general shape of the graph. ~~It is often easier to create a table of values if you rearrange the formula so that rather than being a quotient, it is a product of polynomials: $xy = 4$.~~

x	$\frac{1}{x}$	y $\frac{4}{x}$
-2	$-\frac{1}{2}$	-2
-1	-1	-4
$-\frac{1}{2}$	-2	-8
0	∅	∅
$\frac{1}{2}$	2 ~~undefined~~	8
1	1	4
2	$\frac{1}{2}$	2

$-\frac{1}{2} \times \frac{4}{1} =$



Check your graph using your ~~graphing calculator~~. How do the graphs compare?

Summarize the characteristics of the function using a table.

Characteristic	$y = \frac{4}{x}$
Non-permissible value	$x \neq 0$
Behaviour near non-permissible value	approach asymptote to $\pm\infty$

Non-permissible value	$x \neq 0$
Behaviour near non-permissible value	approach asymptote to $\pm\infty$
End behaviour	extends up/down in all four quadrants
Domain	$\{x \mid x \neq 0, x \in R\}$.
Range	$\{y \mid y \neq 0, y \in R\}$.
Equation of vertical asymptote	$x = 0$
Equation of horizontal asymptote	$y = 0$

To see a similar example, see Example 1 on pages 432–434 of *Pre-Calculus 12*.

Working Example 2: Graph a Rational Function Using Transformations

Graph $y = \frac{3}{x-3} + 2$ using transformations.

Solution

Compare the function $y = \frac{3}{x-3} + 2$ to the form $y = \frac{a}{x-h} + k$ to determine the value of the parameters. Then, describe the effect that each parameter has on the graph of $y = \frac{1}{x}$. Base

$y = \frac{1}{x} \longrightarrow y = \frac{3}{x-3} + 2$

① VE $\not{d}$ 3

3 Right 2 up.

②

If the asymptotes of $y = \frac{1}{x}$ are $x = 0$ and $y = 0$, use the above transformation to determine the asymptotes of $y = \frac{3}{x-3} + 2$. Explain your reasoning. VA $x = 3$ HA $y = 2$

Will the graph of $y = \frac{3}{x-3} + 2$ have an x-intercept or y-intercept? Explain how you know.

Yes b/c of translations

What are the x-intercept and y-intercept?

$y = 0$

$0 = \frac{3}{x-3} + 2$

$-2 = \frac{3}{x-3}$

$-2(x-3) = 3$

$-2x + 6 = 3$

$-2x = -3$

$x = \frac{3}{2}$

$\boxed{(\frac{3}{2}, 0)\ x\text{-int}}$

$y = \frac{3}{0-3} + 2$

$y = \frac{3}{-3} + 2$

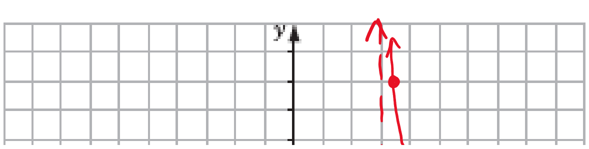
$y = -1 + 2$

$y = 1$ $\boxed{(0,1)\ y\text{-int}}$

Which variable is set to 0 to find the x-intercept? the y-intercept?

$y = 0$ $x = 0$

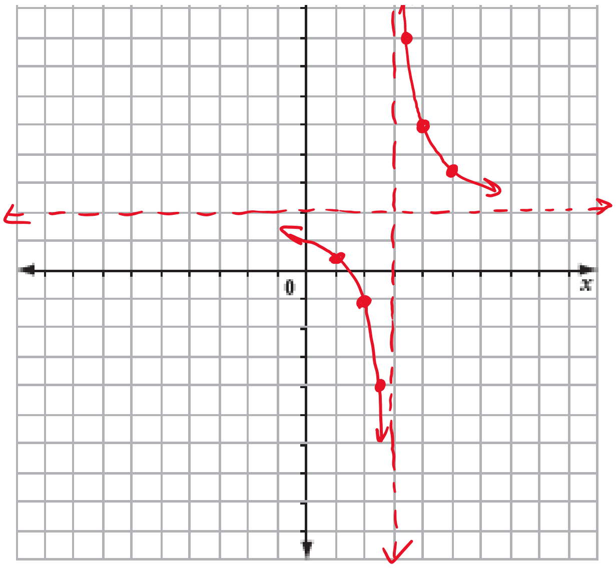
Use all of the above information to graph $y = \frac{3}{x-3} + 2$.



x	y
-2	$-\frac{1}{2}$

x	$3y$
-2	$-\frac{3}{2}$

$x+3$	$3y+2$
1	$\frac{1}{2}$



x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	$\varnothing$
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

x	$3y$
-2	$-\frac{3}{2}$
-1	-3
$-\frac{1}{2}$	-6
0	$\varnothing$
$\frac{1}{2}$	6
1	3
2	$\frac{3}{2}$

$x+3$	$3y+2$
1	$\frac{1}{2}$
2	-1
$\frac{5}{2}$	-4
3	$\varnothing$
$\frac{7}{2}$	8
4	5
5	$\frac{7}{2}$

Check your graph using your graphing calculator. How do the two graphs compare?

NOTE: if you have a horiz. stretch ex $y = \frac{1}{\frac{1}{2}(x+1)} + 3$ combine $\frac{1}{\frac{1}{2}} = 2$

$\boxed{y = \frac{2}{x+1} + 3}$ converted to VE of 2

To see a similar example, see Example 2 on pages 434–435 of *Pre-Calculus 12*.

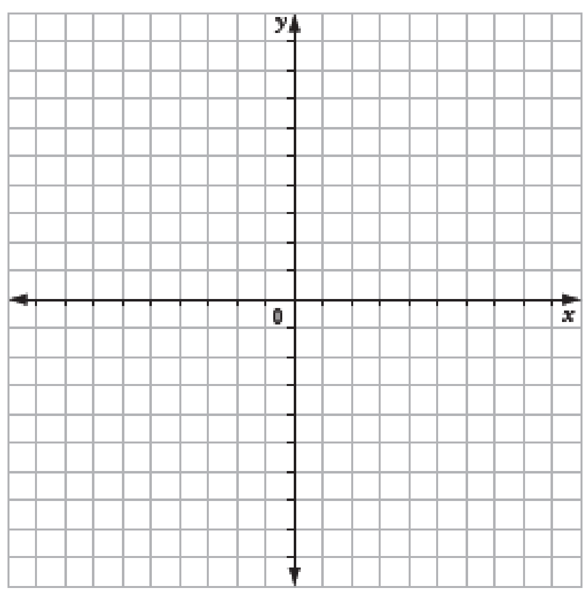
$y = \frac{2}{3(x-5)} + 6$ $a = \frac{2}{3}$ ∴ vs of $\frac{2}{3}$

~~Working Example 3:~~ Graph a Rational Function With Linear Expressions in the Numerator and the Denominator

SKIP

Graph $y = \frac{4x + 2}{x - 1}$. Identify any asymptotes and intercepts.

Solution



Let $x = 0$. Solve for y to determine the y-intercept.

The y-intercept is at (0, ________).

Let $y = 0$. Solve for x to determine the x-intercept.

$$0 = \frac{4x + 2}{x - 1}$$

$$(\_\_\_\_\_\_)(0) = (\_\_\_\_\_\_)\frac{4x + 2}{x - 1}$$

$$\_\_\_\_\_\_ = \_\_\_\_\_\_$$

$$\_\_\_\_\_\_ = 4x$$

$$\_\_\_\_\_\_ = x$$

The x-intercept is at (________, 0).

Manipulate the equation of the function algebraically to obtain the form $y = \frac{a}{x - h} + k$.

Manipulate the equation of the function algebraically to obtain the form $y = \frac{a}{x-h} + k$.

$y = \frac{4x + 2}{x - 1}$

$y = \frac{4x - 4 + 4 + 2}{x - 1}$

$y =$

Why is 4 subtracted and added to the numerator?

Which parameters determine the vertical and horizontal asymptotes of the transformed function?

The parameters are $a =$ ________, $h =$ ________, and $k =$ ________. State the effect of each parameter on the graph of $y = \frac{1}{x}$. Then, use the information you have generated to sketch the transformed function on the grid above.

To see a similar example, see Example 3 on pages 435–437 of *Pre-Calculus 12*.

9.2 - Summary of Rational Graph Characteristics

① Factor + Simplify the Rational equation

② Determine NPV (Restrictions) from denominators in original equation (b/f cancelling)

③ Determine characteristics

$$y = \frac{C(x-a)\cancel{(x-b)}}{(x-p)\cancel{(x-b)}}$$

→ Point of Discontinuity (POD) = HOLE at the NPV $x=b$

- sub $x=b$ into [simplified function] to determine y-coord

$\therefore$ POD $= (x,y)$

ex: (b,y)

$$y = \frac{C(x-a)}{x-p}$$

→ remaining denominator $(x-p)$ is the VA at $x=p$ b/c NPV is $x \neq p$

C is the coefficient which is used to determine the horizontal asymptote (HA)

$$y = \frac{Cx^2 + \ldots}{x^2 + \ldots}$$

= same degree

$\therefore$ HA is $y = C$

$$y = \frac{Cx^2 + \ldots}{x + \ldots}$$

= numerator degree > denominator degree

$\therefore$ no HA or slant asymptote (not in this course)

$$y = \frac{Cx + \ldots}{x^2 + \ldots}$$

= numerator degree < denominator degree

$\therefore$ $y=0$ asymptote but check with graph for exceptions

$$y = \frac{C(x-a)}{x-p}$$

→ x-int is at $x = a$ $\therefore (a,0)$ → b/c $y=0$

$$0 = \frac{C(x-a)}{x-p}$$

$$\frac{0}{C} = \frac{C(x-a)}{C}$$

$$0 = x - a$$

→ y-int make $x=0$ + determine y $\therefore (0,y)$

VA ↓ POD ↓

$$\{x \mid x \neq p, x \neq b, x \in \mathbb{R}\}.$$

$\{x \mid x \neq p, x \neq b, x \in \mathbb{R}\}$.

$\{y \mid y \neq c, y = y, y \in \mathbb{R}\}$.

HA POD

(4) Graph all characteristics.

(5) Use a table of values to fill-in any point to help draw graph.

x	y
?	
?	
?	

Working Ex #1

p. 306.

$$f(x) = \frac{x^2 - 3x - 4}{x - 4}$$

(1) factor + simplify

$$f(x) = \frac{\cancel{(x-4)}(x+1)}{\cancel{x-4}}$$

$$\boxed{f(x) = x+1}$$

(2) NPV's $\Rightarrow x \neq 4$

(3) POD @ $x = 4$

sub into $y = x+1$

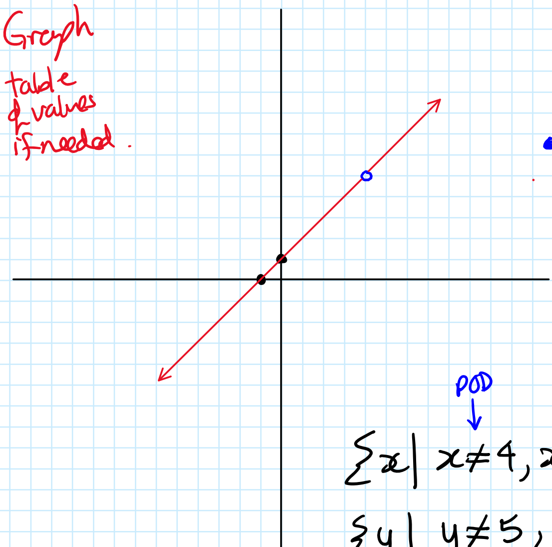
$\therefore y = 4+1$

$y = 5$

$\boxed{POD = (4,5)}$

(4) Graph

(5) table of values if needed.



- no HA, no VA
- x-int $\longrightarrow$ $0 = x+1$, -1, $x = -1$ $\therefore \boxed{(-1,0)}$

y-int $\longrightarrow$ $y = 0+1$, $y = 1$ $\therefore \boxed{(0,1)}$

POD

$\{x \mid x \neq 4, x \in \mathbb{R}\}$.

$\{y \mid y \neq 5, y \in \mathbb{R}\}$.

POD

p.307 $f(x) = \dfrac{x^2 + 4x + 3}{x+1}$ $\qquad$ $g(x) = \dfrac{x^2 - 4x + 3}{x+1}$

① $f(x) = \dfrac{\cancel{(x+1)}(x+3)}{\cancel{x+1}}$

$\boxed{f(x) = x+3}$

POD @ $x = -1$

$y = -1 + 3$

$y = 2$ $\boxed{(-1, 2)}$

x-int $0 = x + 3$

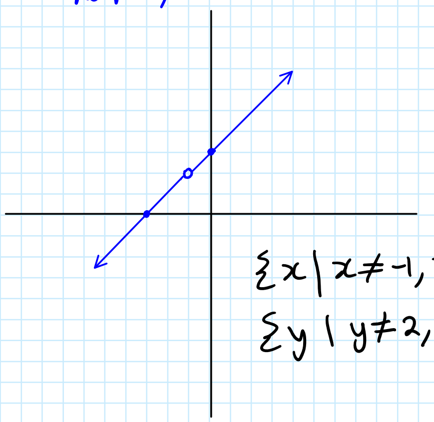
-3

$x = -3$ $\boxed{(-3, 0)}$

y-int $y = 0 + 3$

$y = 3$ $\boxed{(0, 3)}$

no HA, no VA



$\{x \mid x \neq -1, x \in \mathbb{R}\}$

$\{y \mid y \neq 2, y \in \mathbb{R}\}$.

$\boxed{g(x) = \dfrac{(x-1)(x-3)}{x+1}}$

VA @ $x = -1$ $\qquad$ no POD

HA none or slant $\quad$ ☆ check in graph. ✓ no HA

x-int : $x - 1 = 0$ $\qquad$ $x - 3 = 0$

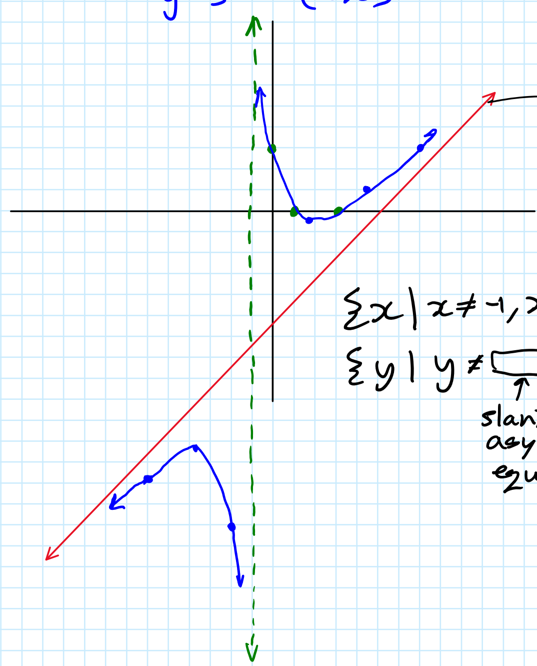
$\therefore$ $x = 1$ $\qquad$ $x = 3$

$(1, 0)$ $\qquad$ $(3, 0)$

y-int : $y = \dfrac{(0-1)(0-3)}{0+1}$

$y = \dfrac{(-1)(-3)}{1}$

$y = 3$ $\qquad$ $(0, 3)$



$\therefore$ no HA

$\{x \mid x \neq -1, x \in \mathbb{R}\}$

$\{y \mid y \neq \square, y \in \mathbb{R}\}$

slant asympt equation

Ex 3 p. 308

$f(x) = \dfrac{x^2 + 2x - 8}{x^2 + 5x + 4}$

① factor + simplify $\quad$ $f(x) = \dfrac{(x-2)\cancel{(x+4)}}{(x+1)\cancel{(x+4)}}$ $\quad$ NPVs. $x \neq -1, -4$

$\boxed{f(x) = \dfrac{1}{1}\dfrac{x-2}{x+1}}$

② NPVs $\qquad$ POD @ $x = -4$ $\qquad$ x-int

$$\boxed{f(x) = \frac{x-2}{x+1}}$$

$x \neq -1, -4$

② NPVs

③ characteristics

④ graph characteristics

⑤ graph with table of values.

POD @ $x=-4$

$y = \frac{-4-2}{-4+1}$

$y = \frac{-6}{-3}$ $\boxed{(-4, 2)}$

$y = 2$

x-int

$\boxed{(2,0)}$

y-int

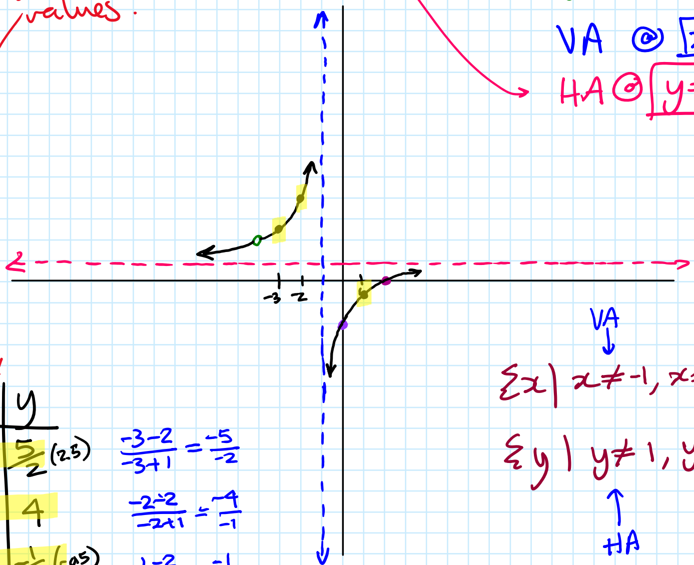
$y = \frac{0-2}{0+1}$

$y = -2$

$\boxed{(0,-2)}$

VA @ $\boxed{x=-1}$

HA @ $\boxed{y=1}$



x	y
-3	$\frac{5}{2}$ (2.5)
-2	4
1	$-\frac{1}{2}$ (-0.5)

$\frac{-3-2}{-3+1} = \frac{-5}{-2}$

$\frac{-2-2}{-2+1} = \frac{-4}{-1}$

$\frac{1-2}{1+1} = \frac{-1}{2}$

$y = \frac{x-2}{x+1}$

VA POD

$\{x \mid x \neq -1, x \neq -4, x \in \mathbb{R}\}$.

$\{y \mid y \neq 1, y \neq 2, y \in \mathbb{R}\}$.

HA POD.

TRY p.311 #4. & #1-5 in ch9 Project.

9.2 Analysing Rational Functions

KEY IDEAS

Determining Asymptotes and Points of Discontinuity

The graph of a rational function may have an asymptote, a point of discontinuity, or both. To establish these important characteristics of a graph, begin by factoring the numerator and denominator fully.

- **Asymptotes: No Common Factors**

  If the numerator and denominator do not have a common factor, the function has an asymptote.

  – The vertical asymptotes are identified by the non-permissible values of the function.

  – For a function that can be rewritten in the form $y = \frac{a}{x-h} + k$, the k parameter identifies the horizontal asymptote.

Example: $y = \frac{x+4}{x-3}$

Since the non-permissible value is $x = 3$, the vertical asymptote is at $x = 3$.

$$y = \frac{x+4}{x-3}$$
$$y = \frac{x-3+3+4}{x-3}$$
$$y = \frac{x-3}{x-3} + \frac{7}{x-3}$$
$$y = \frac{7}{x-3} + 1$$

Method - Skip

do as usual. (as below)

Since $k = 1$, the horizontal asymptote is at $y = 1$.

- **Points of Discontinuity: At Least One Common Factor**

  If the numerator and denominator have at least one common factor, there is at least one point of discontinuity in the graph.

  – Equate the common factor(s) to zero and solve for x to determine the x-coordinate of the point of discontinuity.

  – Substitute the x-value in the simplified expression to find the y-coordinate of the point of discontinuity.

Example: $y = \frac{(x-4)\cancel{(x+2)}}{\cancel{x+2}} \rightarrow \boxed{y = x - 4}$ linear

$x + 2 = 0$: the x-coordinate of the point of discontinuity is -2.

NPV $x \neq -2$

Substitute $x = -2$ into the simplified equation:

$y = x - 4$

$y = -2 - 4$

$y = -6$

$\underbrace{\text{point of discontinuity}}_{\text{POD}}$: $(-2, -6)$ = HOLE in graph

- **Both Asymptote(s) and Point(s) of Discontinuity**

  If a rational expression remains after removing the common factor(s), there may be both a point of discontinuity and asymptotes.

Example:

$$y = \frac{(x-4)\cancel{(x+2)}}{\cancel{(x+2)}(x-1)}$$
$$y = \frac{(x-4)}{(x-1)}$$

NPV $x \neq -2$

– common factor: $x + 2$, so there is a point of discontinuity at $(-2, 2)$

– non-permissible value: $x = 1$, so the vertical asymptote is at $x = 1$

NPV $x \neq 1$

If POD is at $x = -2$

$y = \frac{-2-4}{-2-1}$

$y = -6$ ...

$y = \frac{\quad}{-2-1}$

$y = \frac{-6}{-3}$

$y = 2 \therefore \boxed{POD\ (-2, 2)}$

– non-permissible value: $x = 1$, so the vertical asymptote is at $x = 1$ NPV $x \neq 1$

– simplified function can be rewritten as ~~$y = \frac{3}{x-1}$~~ + 1, so the horizontal asymptote is at $y = 1$ $\boxed{y = \frac{x-4}{x-1}}$

Working Example 1: Graph a Rational Function With a Point of Discontinuity

Sketch the graph of $f(x) = \frac{x^2 - 3x - 4}{x - 4}$.

Solution

Fully factor the numerator and denominator of the rational function.

There is a common factor, so the graph of the function has a ______________________.

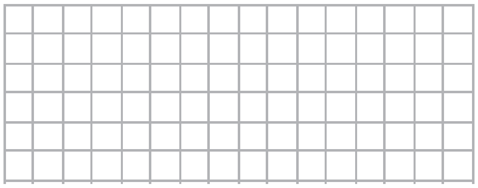
Simplify the rational function. What type of equation remains after the function is simplified?

Equate the common factor to zero and solve for x. Doing so identifies the _______-value of the ______________________.

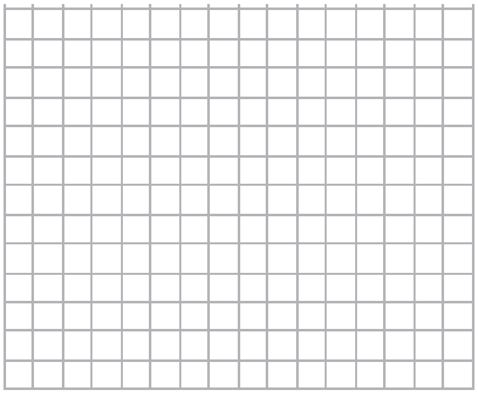
Substitute the value of x into the simplified function and solve for y. Doing so identifies the _______-value of the point of discontinuity in the graph.

The point of discontinuity is _______.

Graph the rational function, labelling the point of discontinuity.



Does it matter if you graph the original equation or the simplified



Does it matter if you graph the original equation or the simplified equation? Explain.

To see a similar example, see Example 1 on pages 447–448 of *Pre-Calculus 12*.

Working Example 2: Compare Points of Discontinuity and Asymptotes in Rational Functions

Compare the graphs of $f(x) = \frac{x^2 + 4x + 3}{x + 1}$ and $g(x) = \frac{x^2 - 4x + 3}{x + 1}$.

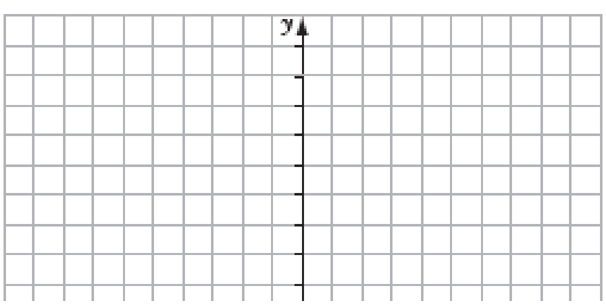
Solution

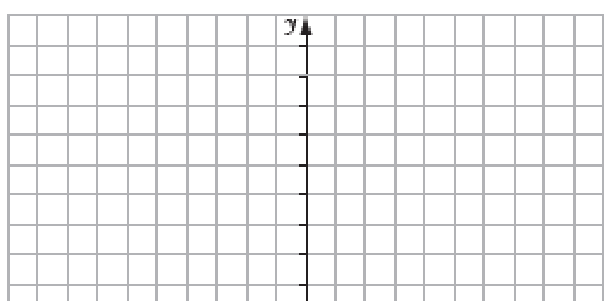
Fully factor the numerator and denominator of each rational function. Simplify the rational functions, if possible. How do the two simplified equations differ?

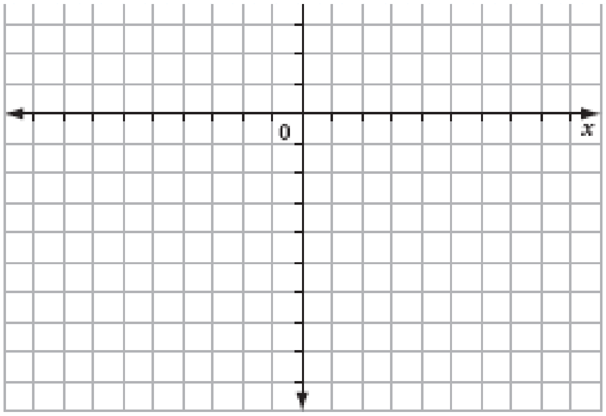
When simplified, $f(x)$ is a ______________ function. It has a(n) ______________.
(*point of discontinuity* or *asymptote*)

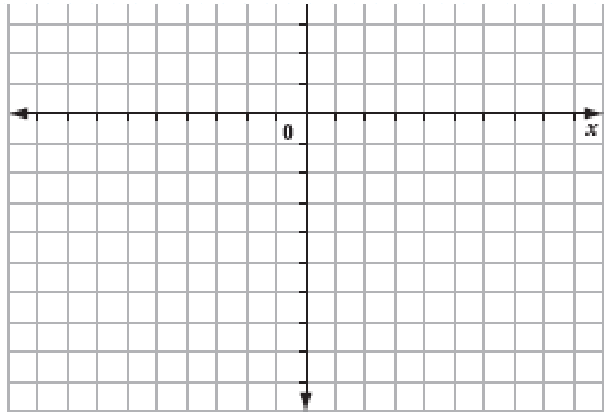
When simplified, $g(x)$ is a ______________ function. It has a(n) ______________.
(*point of discontinuity* or *asymptote*)

With the help of technology, sketch the graph of each function on the grids below. Draw and label any asymptotes that exist.









Both $f(x)$ and $g(x)$ have a non-permissible value at $x =$ ________. Describe what happens to each function as the graph approaches this non-permissible value.

To see a similar example, see Example 2 on pages 448–449 of *Pre-Calculus 12*.

Working Example 3: Sketch a Discontinuous Rational Function

Sketch the graph of $f(x) = \frac{x^2 + 2x - 8}{x^2 + 5x + 4}$. Label all important parts of the graph.

Solution

Fully factor the numerator and denominator of the rational function. Simplify the rational function.

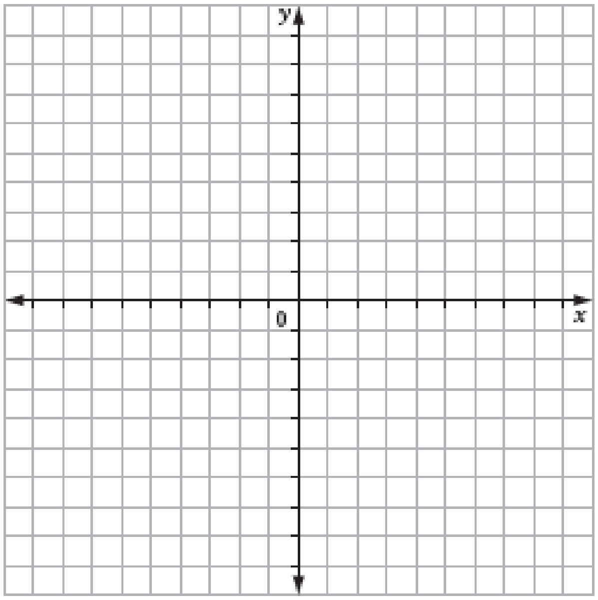
Equate the common factor to zero and solve for x to establish the ____________________ of the point of discontinuity. Substitute the value of x into the simplified function and solve for y to establish the ____________________ of the point of discontinuity in the graph.

The point of discontinuity is at ________.

Find the x-intercept and y-intercept of the simplified function.

Find the x-intercept and y-intercept of the simplified function.

Find the horizontal and vertical asymptotes of the simplified function. Then, use the information you have generated to graph the general shape of the rational function. Label the point of discontinuity, the asymptotes, and the intercepts. Check your sketch using technology.



To see a similar example, see Graph 2 in Example 3 on page 449 of *Pre-Calculus 12*.

9.2: Analysing Rational Function Graphs

Graph Rational Functions with Holes

If the degree of the numerator < degree of the denominator then horizontal asymptote is at y = 0.

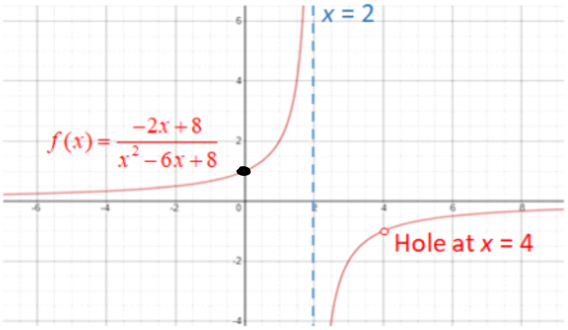
The vertical asymptotes will occur where the denominator equals zero.

If there is a common factor in the numerator and denominator then the graph of a rational function will have a hole when a value of x causes both the numerator and the denominator to equal 0. We can set the common factor to zero and solve for x to find the hole.

Example:

$$f(x) = \frac{-2x+8}{x^2-6x+8}$$
$$= \frac{-2(x-4)}{(x-4)(x-2)}$$

hole at $x = 4$ vertical asymptote at $x = 2$



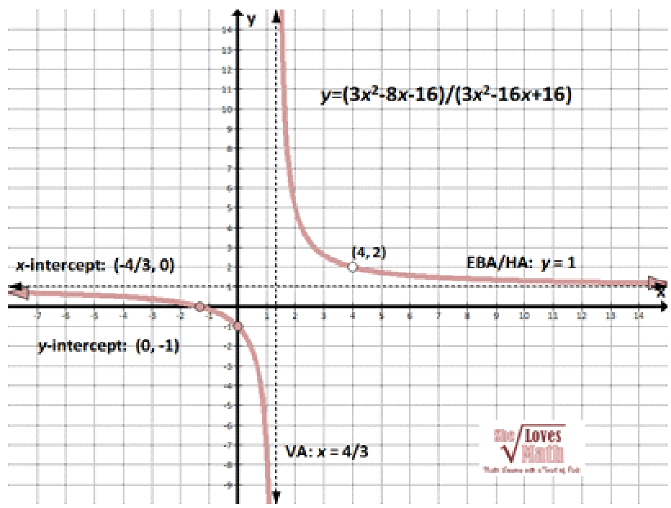
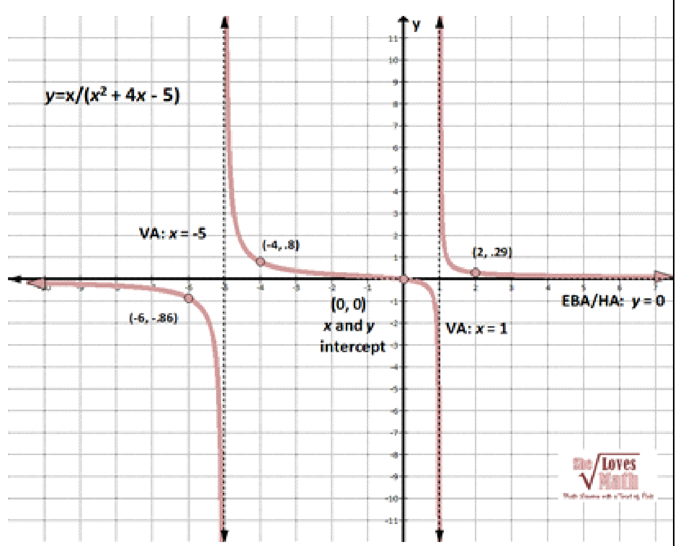
Sketching the Graph of a Rational Function by Hand

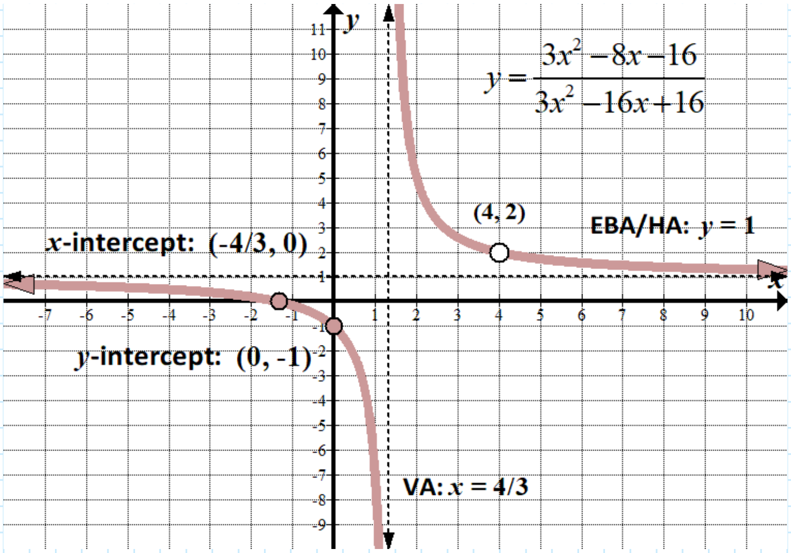
Guidelines for Graphing Rational Functions

1. Write the rational expression in **simplest form**, by factoring the numerator and denominator and dividing out common factors.
2. Find the coordinates of any **"holes"** in the graph.
3. Find and plot the **y-intercept**, if any, by evaluating **f(0).**
4. Find and plot the **x-intercept(s)**, if any, by **finding the zeros of the numerator**.
5. Find the **vertical asymptote(s)**, if any, by **finding the zeros of the denominator**. Sketch these using dashed lines.
6. Find the **horizontal asymptote**, if any, by **comparing the degrees of the numerator and denominator**. Sketch these using dashed lines.
7. Find the **oblique asymptote**, if any, by **dividing the numerator by the denominator using long division**.
8. Plot **5-10 additional points**, including points close to each x-intercept and vertical asymptote.
9. Use **smooth curves** to complete the graph.

Rules for Graphing Rationals	Examples
To get the **end behavior asymptote (EBA)**, you want to compare the degree in the numerator to the degree in the denominator. There can be **at most 1 EBA** and most of the time, these are horizontal. ➢ If the degree (largest exponent) on the **bottom is greater** than the degree on the top, the EBA (which is also a **horizontal asymptote** or **HA**) is $\mathbf{y=0}$.	$y=\dfrac{x+2}{x^2-4}$ Notice that even though we can take out a removable discontinuity $(x+2)$, the bottom still has a higher degree than the top, so the HA/EBA is $\mathbf{y=0}$.
➢ If the degree on the **top is greater** than the degree on the bottom, there is no EBA/HA. **However, if the degree on the top is one more than the degree on the bottom, than there is a slant (oblique) EBA asymptote, which is discussed below.**	$y=\dfrac{x^3+2}{x-4}$ No HA/EBA. Vertical asymptote is still $\mathbf{x=4}$.
➢ If the degree is the **same on the top and the bottom**, than **divide coefficients of the variables with the highest degree on the top and bottom**; this is the HA/EBA. You can determine this asymptote even without factoring.	$y=\dfrac{2x^3+2}{3x^3-4}$ Since the degree on the top and bottom are both 3, the HA/EBA is $\mathbf{y=\frac{2}{3}}$.
➢ If the degree on the top is **one more** than the degree on the bottom, then the function has a **slant** or **oblique EBA** in the form $y=mx+b$. We have to use **long division** to find this equation. We can just ignore or "throw away" the remainder and just use the linear equation. Weird, huh?	$y=\dfrac{2x^2+x+1}{x-4}$ $\begin{array}{r r l} & 2x+9 & \\ x-4 \big) & 2x^2+x+1 & \\ & 2x^2-8x & \\ & 9x+1 & \\ & 9x-36 & \\ & \cancel{37} & \end{array}$ EBA: $\mathbf{y=2x+9}$
➢ (more Advanced) Find the point where any horizontal asymptotes **cross the function** by **setting the function to the horizontal asymptote**, and solving for "$\boldsymbol{x}$". You already have the "$\boldsymbol{y}$" (from the HA equation).	**Q:** Where does $y=\dfrac{-x^2+x}{x^2+x-12}$ **intersect** its EBA? **A:** Note that the EBA is $y=\dfrac{-1}{1}=-1$. Now set $\dfrac{-x^2+x}{x^2+x-12}=-1$ and cross multiply: $-x^2+x=-1\left(x^2+x-12\right)$; $\mathbf{x=6}$. So the point where the function intersects the EBA is $\mathbf{(6,-1)}$.

Rules for Graphing Rationals	Examples
First factor both the numerator and denominator, and **cross out any factors** in both the numerator and denominator. ➢ If any of these factors contain variables, these are **removable discontinuities**, or "**holes**" and will be little circles on the graphs. The idea is that if you cross out a polynomial, you can't forget that it was in the denominator and can't "legally" be set to 0. (We will see graph later.) The **domain** of a rational function is all real numbers, except those that make the denominator equal zero, as we saw earlier. (Note that if after you cross out factors, you still have that same factor on the bottom, the "hole" will turn into a **vertical asymptote**; follow the rules below).	$y=\frac{x^2-5x+6}{x-3}=\frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}}=x-2$ This function reduces to the line $y=x-2$ with a **removable discontinuity** (a little circle on the graph) where $x=2$ and $y=(2)-2=0$ (plug 2 in for ***y*** in original or reduced fraction). So the hole is at (2, 0). Domain is $(-\infty,3)\cup(3,\infty)$, since a 3 would make the denominator = 0. It's like we have to "skip over" the 3 with interval notation.
To get **vertical asymptotes or VAs**: ➢ After determining if there are any holes in the graph, factor (if necessary) what's left in the **denominator** and set the factors to 0. For any value of ***x*** where these factors could be 0, this creates a vertical asymptote at "***x* =** " for these values. Note: There could a multiple number of vertical asymptotes, or no vertical asymptotes. Don't forget to include the factors with "***x***" alone (***x* = 0** is the vertical asymptote).	$y=\frac{x^2-5x+6}{x(x^2-9)}=\frac{\cancel{(x-3)}(x-2)}{x\cancel{(x-3)}(x+3)}=\frac{x-2}{x(x+3)}$ Vertical asymptotes occur when $(x-0)=0$ or $(x+3)=0$, or $x=0$ or $x=-3$. Domain is $(-\infty,-3)\cup(-3,0)\cup(0,3)\cup(3,\infty)$, since anything that could make the denominator 0 (even a hole) can't be included. So we have to "skip over" –3, 0, and 3.

Steps		Graph
$y=\dfrac{3x^2-8x-16}{3x^2-16x+16}$ Factor: $y=\dfrac{(3x+4)\cancel{(x-4)}}{(3x-4)\cancel{(x-4)}}$ **Removable Discontinuity or Hole:** ***x*** = 4, plug in 4 for ***x*** to get ***y*** = 2. RD is (4, 2). **VA:** Set denominator to 0 after removing the hole: $3x-4=0 \quad x=\frac{4}{3}$.	**EBA/HA:** Since the degree is the same on the top and bottom (both are 2), we take the coefficients and divide them: $y=\frac{3}{3}=1$. ***x*-intercept (root):** Set ***y*** (or top) to 0: $3x+4=0; \quad x=-\frac{4}{3} \quad \left(-\frac{4}{3},0\right)$ ***y*-Intercept:** Set ***x*** to 0: $y=\dfrac{3(0)+4}{3(0)-4}=-1 \quad (0,-1)$ **Domain:** Can't be any value of ***x*** that makes the bottom zero: $(-\infty,\frac{4}{3})\cup(\frac{4}{3},4)\cup(4,\infty)$	
$y=\dfrac{x}{x^2+4x-5}$ Factor: $y=\dfrac{x}{(x+5)(x-1)}$ **VA:** Set denominator to 0 after factoring; we have 2 of them: $x=-5, \quad x=1$. **EBA/HA:** Since the degree on the top (1) is less than the degree on the bottom (2), the EBA or VA is ***y*** = 0.	***x*-intercept (root):** Set ***y*** (or top) to 0: $x=0. \quad (0,0)$ ***y*-Intercept:** Set ***x*** to 0: $y=\dfrac{(0)}{(0)^2+4(0)-5}=0: \; (0,0)$ **"T-chart"**: Try some points around the vertical asymptotes: $x=-6, \quad y=-.86$ $x=-4, \quad y=.8$ $x=2, \quad y=.29$ **Domain:** Can't be any value of ***x*** that makes the bottom zero: $(-\infty,-5)\cup(-5,1)\cup(1,\infty)$.	



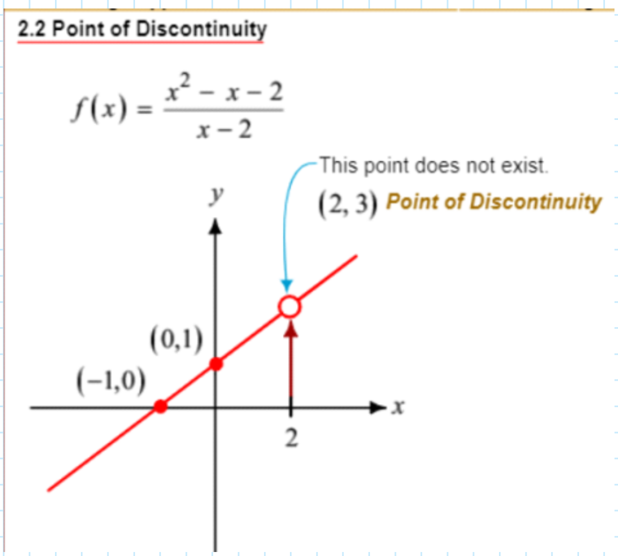
Rules for Asymptotes

Exponent of highest degree term in denominator larger than highest degree term in numerator then Horizontal Asymptote is

$y = 0$

Exponent values of highest degree of terms in numerator and denominator the same then Horizontal Asymptote is

$y =$ ratio of their coefficients

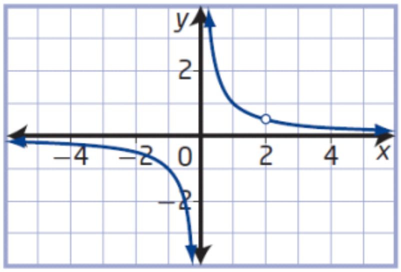


5. Which graph matches each rational function? Explain your choices.

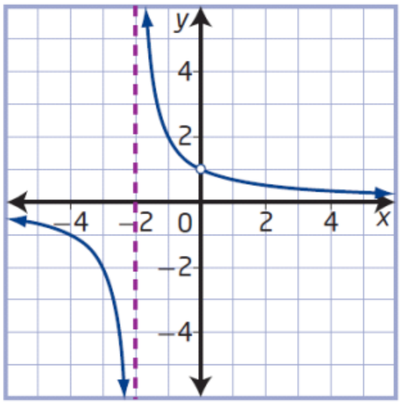
a) $A(x) = \dfrac{x^2 + 2x}{x^2 + 4}$ **b)** $B(x) = \dfrac{x - 2}{x^2 - 2x}$

c) $C(x) = \dfrac{x + 2}{x^2 - 4}$ **d)** $D(x) = \dfrac{2x}{x^2 + 2x}$

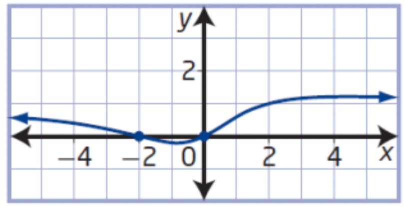
A



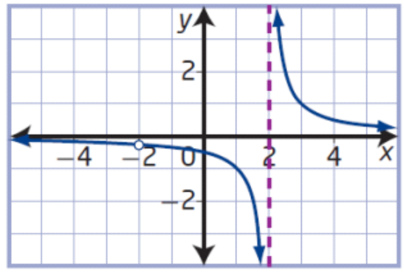
B



C



D



8. Write the equation of a possible rational function with each set of characteristics.

a) vertical asymptotes at $x = \pm 5$ and x-intercepts of -10 and 4

b) a vertical asymptote at $x = -4$, a point of discontinuity at $\left(-\frac{11}{2}, 9\right)$, and an x-intercept of 8

c) a point of discontinuity at $\left(-2, \frac{1}{5}\right)$, a vertical asymptote at $x = 3$, and an x-intercept of -1

d) vertical asymptotes at $x = 3$ and $x = \frac{6}{7}$, and x-intercepts of $-\frac{1}{4}$ and 0